Foldy-Wouthuysen transformation for relativistic particles in external fields

Alexander J. Silenko

Institute of Nuclear Problems, Belarusian State University,

Bobruiskaya str., 11, 220080 Minsk, Belarus

(Dated: February 5, 2008)

Abstract

A method of Foldy-Wouthuysen transformation for relativistic spin-1/2 particles in external fields is proposed. It permits determination of the Hamilton operator in the Foldy-Wouthuysen representation with any accuracy. Interactions between a particle having an anomalous magnetic moment and nonstationary electromagnetic and electroweak fields are investigated.

PACS numbers: 03.65.Pm, 11.10.Ef, 12.20.Ds

I. INTRODUCTION

The Foldy-Wouthuysen (FW) representation [1] occupies special place in the quantum theory. This is mainly due to the fact that the Hamiltonian and all operators in this representation are block-diagonal (diagonal in two spinors). For relativistic particles in external fields, operators have the same form as in the nonrelativistic quantum theory. Therefore, the FW representation in the relativistic quantum theory is similar to the nonrelativistic quantum theory. The basic advantages of the FW representation are described in [1, 2] (see also below).

The transformation to the FW representation (FW transformation) holds only in the one-particle approximation where the radiative corrections are not calculated in a consistent way but are phenomenologically taken into account by including extra terms in the Dirac equation (see [3]). One-particle description is feasible even for ultrarelativistic particles if the external field is so weak that the probability of pair production or bremsstrahlung losses can be neglected for a given interaction energy of a particle. The range of applicability of this description is fairly wide and includes, in particular, the relativistic particle scattering and the interaction of relativistic particles with matter and external fields.

In the nonrelativistic case, there exist a lot of good methods of FW transformation with taking into account relativistic corrections [1, 4, 5, 6]. However, they are not useful for relativistic particles. The known methods of solving this problem [9, 10, 11, 12] either lead to cumbersome calculations or the field of their use is limited by the first approximation in field parameters. None of these methods permits exact FW transformation for the particular cases described in [5, 7, 8]. Therefore, the FW representation does not take the right stand in the relativistic quantum theory. The Dirac and Melosh [13] representations are mostly used.

FW transformation can also be performed for particles with spin $s > 1/2$ [7, 14].

In the present work, a method of FW transformation for relativistic particles in external fields is proposed. This method permits obtaining a Hamiltonian of any accuracy by successive approximations, as a power series in the external field potentials and their derivatives. In some cases, this method permits performing exact FW transformation.

The relativistic system of units $\hbar = c = 1$ is used.

II. GENERAL PROPERTIES OF THE FOLDY-WOUTHUYSEN REPRESENTATION

The basic advantages of the FW representation are due to its specific properties.

The relations between the operators in the FW representation are similar to those between the respective classical quantities. In this representation, the operators have the same form as in the nonrelativistic quantum theory. Only the FW representation possesses these properties considerably simplifying the transition to the semiclassical description. The FW representation provides the best possibility of obtaining a meaningful classical limit of the relativistic quantum theory.

For example, the Hamiltonian for a free particle fully agrees with that of classical physics:

$$\mathcal{H}_{FW} = \beta\sqrt{m^2+\boldsymbol{p}^2}, \quad \boldsymbol{p} = -i\nabla, \tag{1}$$

in contrast with the Hamiltonian in the Dirac representation [1, 15]. The position operator in the Dirac representation is the radius-vector, $\boldsymbol{r}$ [15]. It corresponds to the mean position operator for the free particle in the FW representation [1],

$$\boldsymbol{r}_D = \boldsymbol{r} + \frac{i\beta\boldsymbol{\alpha}}{2\epsilon} - \frac{i\beta(\boldsymbol{\alpha}\cdot\boldsymbol{p})\boldsymbol{p} + [\boldsymbol{\Sigma}\times\boldsymbol{p}]p}{2\epsilon(\epsilon+m)p}, \quad p \equiv |\boldsymbol{p}|, \quad \epsilon = \sqrt{m^2+p^2}.$$

Here and below the following designations for the matrices are used:

$$\boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}, \quad \beta \equiv \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \boldsymbol{\alpha} = \beta\boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix},$$
$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}, \quad \boldsymbol{\Pi} = \beta\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & -\boldsymbol{\sigma} \end{pmatrix},$$

where $0, 1, -1$ mean the corresponding 2×2 matrices and $\boldsymbol{\sigma}$ is the Pauli matrix.

In the FW representation, the problem of "*zitterbewegung*" motion never arises [2, 15]. The operators $\boldsymbol{l} = \boldsymbol{r}\times\boldsymbol{p}$ and $\boldsymbol{\Sigma}/2$ define the angular momentum and the spin for the free particle, respectively. In this representation, unlike the Dirac one, each of them is a constant of motion (see [1]). The corresponding operators conserving in the Dirac representation are

$$\boldsymbol{l}_D = \boldsymbol{r}_D\times\boldsymbol{p},$$
$$\frac{\boldsymbol{\Sigma}_D}{2} = \frac{\boldsymbol{\Sigma}}{2} - \frac{i\beta[\boldsymbol{\alpha}\times\boldsymbol{p}]}{2\epsilon} - \frac{[\boldsymbol{p}\times[\boldsymbol{\Sigma}\times\boldsymbol{p}]]}{2\epsilon(\epsilon+m)}.$$

The total angular momentum operator, $\boldsymbol{j}$, is a constant of motion in both representations, because

$$\boldsymbol{j}_D=\boldsymbol{l}_D+\frac{\boldsymbol{\Sigma}_D}{2}=\boldsymbol{l}+\frac{\boldsymbol{\Sigma}}{2}=\boldsymbol{j}.$$

The FW representation is very convenient for describing the particle polarization. In this representation, polarization operators have simple forms. For example, the three-dimensional polarization operator equals the matrix $\boldsymbol{\Pi}$ [16, 17]. In the Dirac representation, this operator depends on the particle momentum [16, 17]:

$$\boldsymbol{O}\equiv\boldsymbol{\Pi_D}=\boldsymbol{\Pi}-\gamma^5\frac{\boldsymbol{p}}{\epsilon}-\frac{\boldsymbol{p}(\boldsymbol{\Pi}\cdot\boldsymbol{p})}{\epsilon(\epsilon+m)}.$$

For particles interacting with external fields, it also depends on the external field parameters [17].

Thus, in the Dirac representation all operators corresponding to the basic classical quantities are defined by cumbersome expressions. These operators should also depend on the external field parameters for particles interacting with external fields.

The FW representation helps one to prove that the particle position can be measured up to its Compton wavelength [1, 15]. However, this property is valid only for a particle not strongly interacting with external fields if the one-particle approximation is attainable. Otherwise, the effect of pair production prevents the use of both the Dirac equation (even with some corrections) and the "traditional" Hamilton approach. Obviously, in this case the FW transformation cannot be used either.

The FW transformation possesses another important property. The relativistic wave equations and all operators are block-diagonal (diagonal in two spinors). This property permits separating positive and negative energy states [1]. Of course, extraction of even parts of operators becomes unnecessary.

The detailed analysis performed in [18] shows that the wave functions in both the Dirac and FW representations are equal to each other only approximately, and they do not coincide. In these papers, the nonrelativistic case was considered and relativistic corrections was taken into account.

An analogous conclusion follows from the results obtained in [11]. In this work, a more general situation has been investigated for the relativistic particle not strongly interacting with an electromagnetic field. It has been found that the upper spinors in the Dirac and

FW representations are approximately proportional to each other, but this property is not exact.

Thus, the preferable employment of the FW representation is evident, although the relativistic wave equations are more complicated in this representation.

III. METHODS OF THE FOLDY-WOUTHUYSEN TRANSFORMATION

In the classical work by Foldy and Wouthuysen [1], two different transformations, for free relativistic particles and for nonrelativistic particles in electromagnetic fields have been carried out. In the general case, transformation to a new representation described by the wave function Ψ' is performed with the unitary operator U:

$$\Psi' = U\Psi = e^{iS}\Psi,$$

where $\Psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}$ is the wave function (bispinor) in the Dirac representation. As

$$\Psi = U^{-1}\Psi', \quad i\frac{\partial}{\partial t}\Psi = \mathcal{H}\Psi, \quad i\frac{\partial}{\partial t}\Psi' = \mathcal{H}'\Psi',$$

the following transformation can be carried out:

$$\begin{aligned} i\frac{\partial}{\partial t}\Psi = \mathcal{H}U^{-1}\Psi', \quad i\frac{\partial}{\partial t}\Psi = i\frac{\partial}{\partial t}\left(U^{-1}\Psi'\right) = i\frac{\partial U^{-1}}{\partial t}\Psi' + iU^{-1}\frac{\partial \Psi'}{\partial t} \\ = \left(i\frac{\partial U^{-1}}{\partial t} + U^{-1}\mathcal{H}'\right)\Psi', \quad U\mathcal{H}U^{-1}\Psi' = \left(iU\frac{\partial U^{-1}}{\partial t} + \mathcal{H}'\right)\Psi'. \end{aligned}$$

Hence, the Hamilton operator in the new representation takes the form [1, 19]:

$$\mathcal{H}' = U\mathcal{H}U^{-1} - iU\frac{\partial U^{-1}}{\partial t}, \tag{2}$$

or

$$\mathcal{H}' = U\left(\mathcal{H} - i\frac{\partial}{\partial t}\right)U^{-1} + i\frac{\partial}{\partial t}.$$

There is an error in this transformation in [20].

The Hamiltonian can be split into operators commuting and noncommuting with the operator β:

$$\mathcal{H} = \beta m + \mathcal{E} + \mathcal{O}, \quad \beta\mathcal{E} = \mathcal{E}\beta, \quad \beta\mathcal{O} = -\mathcal{O}\beta. \tag{3}$$

The Hamiltonian $\mathcal{H}$ is Hermitian. We assume that both operators $\mathcal{E}$ and $\mathcal{O}$ are also Hermitian.

For free Dirac particles $\mathcal{E} = 0$, $\mathcal{O} = \boldsymbol{\alpha} \cdot \boldsymbol{p}$, and the operator S has the form

$$S = -i\beta\boldsymbol{\alpha} \cdot \boldsymbol{p}\theta(\boldsymbol{p}), \tag{4}$$

where θ is a function of the momentum operator. If we choose

$$\theta(\boldsymbol{p}) = \frac{1}{2p} \arctan\left(\frac{p}{m}\right),$$

the transformed Hamiltonian $\mathcal{H}'$ contains no odd operators [1, 20] and we obtain Eq. (1)

$$\mathcal{H}' = \beta\sqrt{m^2 + \boldsymbol{p}^2}.$$

For nonrelativistic particles in an electromagnetic field, the FW transformation can be performed with the operator [1, 20]

$$S = -\frac{i}{2m}\beta\mathcal{O}. \tag{5}$$

The transformed Hamiltonian can be written in the form

$$\begin{aligned} \mathcal{H}' = \mathcal{H} + i[S, \mathcal{H}] + \frac{i^2}{2!}[S, [S, \mathcal{H}]] + \frac{i^3}{3!}[S, [S, [S, \mathcal{H}]]] + \dots \\ -\dot{S} - \frac{i}{2!}[S, \dot{S}] - \frac{i^2}{3!}[S, [S, \dot{S}]] - \dots, \end{aligned} \tag{6}$$

where $[\dots, \dots]$ means a commutator. As a result of this transformation, we find

$$\mathcal{H}' = \beta\epsilon + \mathcal{E}' + \mathcal{O}', \quad \beta\mathcal{E}' = \mathcal{E}'\beta, \quad \beta\mathcal{O}' = -\mathcal{O}'\beta, \tag{7}$$

where the odd operator $\mathcal{O}'$ is now $O(1/m)$. This procedure can be repeated to obtain the required accuracy. Another form of the nonrelativistic FW transformation was given by Ericksen [5] (see also [21]).

There are also other methods for obtaining the block-diagonal form of the Hamiltonian or the Lagrangian. The so-called elimination method of Pauli [22] permits excluding the lower spinor from relativistic wave equations. As a result, the wave function of the final Pauli equation is the upper Dirac spinor, ϕ. This means that the upper Dirac spinor is also an eigenfunction of the transformed Hamiltonian. However, this property is not exact. The Pauli method was analyzed in detail in [18, 21]. It was shown that this method gives the right first approximation. Nevertheless, relativistic corrections of higher orders are incorrect. It

is quite natural because direct Pauli's reduction leads to a neglect of the contribution of the lower spinor [18]. The relation between the exact wave function in the FW representation and the upper Dirac spinor has been found in [11] in the relativistic case.

A more exact variant of the elimination method had been proposed earlier by Berestetskii and Landau [23] (see also [4, 24]). They showed that it was possible to find a nonunitary operator V for which

$$\psi = V\phi \tag{8}$$

is a two-component wave function with a correct norm. An appropriate form of the operator V can be obtained from the condition

$$\int \psi^\dagger \psi dV = \int (\phi^\dagger \phi + \chi^\dagger \chi) dV = 1.$$

The relation between the Dirac spinors can be expressed in the general form: $\chi = Q\phi$. Therefore,

$$V^\dagger V = 1 + Q^\dagger Q.$$

If we additionally assume that the operator V is Hermitian, then both this operator and the Hamiltonian can be found by successive approximations [4, 23, 24, 25].

Of course, the elimination method is much simpler. However, it is mostly intuitive. Its validity is proved only by the coincidence of the results obtained by the FW and Akhiezer-Berestetskii-Landau methods [21].

Another method of diagonalization of relativistic wave equations was proposed by Korner and Thompson [6]. In this work, the Lagrangian approach was used. The Korner-Thompson method is similar to the FW method. It also includes a successive decrease in the maximum order of odd terms. The results obtained by the FW and Korner-Thompson methods agree (see [12]).

Thus, several nonrelativistic transformation methods give the same results. However, the FW transformation method has been justified in the best way.

In several cases, FW transformation can be performed exactly [5, 7, 8]. Exact FW transformation has also been performed for a wide class of external fields in [26]. In this work, involutive symmetries of relativistic wave equations have been used. However, the transformed Hamiltonians contain "nontraditional" space reflection operators. The reduction of Hamiltonians to the "traditional" form is a difficult problem. It has not been investigated

in [26]. However, this reduction is necessary to do for solving many problems (e.g., finding particle and spin motion equations).

Generally, FW transformation for relativistic particles in external fields is complicated. The transformation methods explained in [9, 10] require cumbersome calculations. A variant of the elimination method useful for relativistic particles has been developed in [11]. On eliminating the lower spinor from the relativistic wave equations, the final equation for the upper spinor takes the form [11]:

$$i\frac{\partial\phi}{\partial t}=F(\boldsymbol{r},\boldsymbol{p},i\frac{\partial}{\partial t})\phi, \tag{9}$$

where F is the operator function. Further calculations are analogous to those in the Akhiezer-Berestetskii-Landau method. A new wave function with a correct norm, ψ, expressed by Eq. (8) is introduced. Substituting it for ϕ into Eq. (9), one can find the Hamilton operator for the relativistic particle.

The relativistic wave equation for an upper spinor similar to Eq. (9) is found by the Lagrangian approach [12].

However, it is difficult to find a second approximation by using the relativistic variant of the elimination method proposed in [11]. It is easier to determine relativistic corrections of higher orders [12].

The right two-component wave function in the FW representation, ψ, does not coincide with the upper Dirac spinor, ϕ [11]. This conclusion is in agreement with the results obtained in [18].

There are other difficult problems. The diagonalization of relativistic wave equations needs carefulness, especially in the time-dependent case. As mentioned in [19, 27], in the latter case $\mathcal{H}'$ is not equivalent to $\mathcal{H}$ since these operators have different matrix elements. Rather, $U\mathcal{H}U^{-1}$ is. There is a danger that one can arrive at a block-diagonal representation differing from the FW one even in the time-independent case. For example, the transformation performed in [8] (this is the Melosh transformation indeed [28]) leads to a block-diagonal Hamiltonian that differs from the Hamiltonian in the FW representation [29]. Therefore, the application of noncanonical transformation methods is restricted by the necessity of verifying the results by comparing them with the corresponding results obtained by the canonical transformation method in some particular cases. Of course, other transformation methods may be simpler or less cumbersome. Nevertheless, the FW method is safer and

substantiated very well.

In the present work, a relativistic extension to the FW method is proposed.

IV. EXACT FOLDY-WOUTHUYSEN TRANSFORMATION

Consider some cases of the exact FW transformation.

In Eq. (3), the operators β and $\mathcal{O}^2$ commute ($\beta\mathcal{O}^2 = -\mathcal{O}\beta\mathcal{O} = \mathcal{O}^2\beta$). Therefore, the operator $\mathcal{O}^2$ is even.

The operator S can be defined by an expression similar to Eq. (4):

$$S = -i\frac{\beta\mathcal{O}}{C}\theta, \tag{10}$$

where C and θ are the functions of $\mathcal{O}^2$ and the operator C satisfies the following conditions:

$$C^2 = \mathcal{O}^2, \qquad [\beta, C] = 0. \tag{11}$$

It follows from conditions (11) that the operator C is also even.

It is possible to use the following formal definition of this operator:

$$C = \sqrt{\mathcal{O}^2}. \tag{12}$$

Relations (11),(12) define the square root of matrix operators. To unambiguously define the square root, these relations should be complemented by the condition that the square root of the unit matrix $\mathcal{I}$ is equal to the unit matrix. This definition of the square root coincides with those of [1, 20, 26]. For example, for free particles

$$\mathcal{O} = \boldsymbol{\alpha}\cdot\boldsymbol{p}, \quad \mathcal{O}^2 = \mathcal{I}\boldsymbol{p}^2, \quad C = \mathcal{I}\sqrt{\boldsymbol{p}^2} \equiv \mathcal{I}|\boldsymbol{p}|.$$

Further, the symbol of the unit matrix $\mathcal{I}$ will be omitted.

Since

$$\mathcal{O}^2 = -\beta\mathcal{O}\beta\mathcal{O}, \quad C = \sqrt{-\beta\mathcal{O}\beta\mathcal{O}}, \quad f(\mathcal{O}^2) = f(C^2),$$

the operators $\beta\mathcal{O}, C, \mathcal{O}^2$, and θ commute with each other. The operator θ is the angle of rotation of the basic vector set in the spinor space.

The unitary transformation operator, U, can be written in the form

$$U = \cos\theta + \frac{\beta\mathcal{O}}{C}\sin\theta. \tag{13}$$

An FW transformation is exact if the external field is stationary and the operators $\mathcal{E}$ and $\mathcal{O}$ commute:

$$[\mathcal{E}, \mathcal{O}] = 0. \tag{14}$$

In this particular case,

$$[\mathcal{E}, \beta\mathcal{O}] = \beta[\mathcal{E}, \mathcal{O}] = 0.$$

Condition (14) is a sufficient but not necessary condition of the exact transformation.

The Hamilton operator in the new representation takes the form

$$\mathcal{H}' = \left(\cos\theta + \frac{\beta\mathcal{O}}{C}\sin\theta\right)\mathcal{H}\left(\cos\theta - \frac{\beta\mathcal{O}}{C}\sin\theta\right)$$
$$= (\beta m + \mathcal{O})\left(\cos\theta - \frac{\beta\mathcal{O}}{C}\sin\theta\right)^2 + \mathcal{E} = (\beta m + \mathcal{O})\left(\cos 2\theta - \frac{\beta\mathcal{O}}{C}\sin 2\theta\right) + \mathcal{E}$$
$$= \beta\left(m\cos 2\theta + C\sin 2\theta\right) + \mathcal{O}\left(\cos 2\theta - \frac{m}{C}\sin 2\theta\right) + \mathcal{E}.$$

The Hamiltonian $\mathcal{H}'$ is even if the odd term (proportional to $\mathcal{O}$) vanishes. This takes place if

$$\tan 2\theta = \frac{C}{m}. \tag{15}$$

This equation has two solutions, θ_1 and θ_2, differing in $\pi/2$. Since

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}, \qquad \tan\theta = \frac{\tan 2\theta}{1 \pm \sqrt{1 + \tan^2 2\theta}},$$

they are defined by the relations

$$\tan\theta_1 = \frac{C}{\epsilon + m}, \qquad \tan\theta_2 = -\frac{C}{\epsilon - m}, \qquad \epsilon = \sqrt{m^2 + C^2} = \sqrt{m^2 + \mathcal{O}^2}. \tag{16}$$

Thus, there are two unitary transformations of the operator $\mathcal{H}$ to an even form. They are characterized by the angles θ_1 and θ_2, where the angle θ_1 corresponds to the FW transformation.

As a result of both transformations, one of the spinors (lower for θ_1 and upper for θ_2) becomes zero as for free particles.

Note that the transformation under consideration is also similar to the Melosh transformation [13].

If condition (14) is satisfied, then the Hamilton operator in the FW representation is defined exactly:

$$\mathcal{H}' = \beta\epsilon + \mathcal{E}. \tag{17}$$

Unlike [9, 10, 11], Eq. (17) contains exact expressions for the Hamiltonian derived in [5, 7, 8] as particular cases.

The transformation operator U can be written in nonexponential form. After the calculation of $\sin\theta_1$ and $\cos\theta_1$ with formulae (16),

$$\sin\theta_1 = \frac{C}{\sqrt{2\epsilon(\epsilon+m)}}, \qquad \cos\theta_1 = \sqrt{\frac{\epsilon+m}{2\epsilon}},$$

we obtain the following expression:

$$U^{\pm} = \frac{\epsilon + m \pm \beta\mathcal{O}}{\sqrt{2\epsilon(\epsilon+m)}}. \tag{18}$$

where $U^+ \equiv U$, $U^- \equiv U^{-1}$. This expression agrees with the well-known formula for free particles [1]. Since $(\beta\mathcal{O})^\dagger = \mathcal{O}\beta = -\beta\mathcal{O}$, the operator U is unitary. Simultaneous change of signs of $\sin\theta_1$ and $\cos\theta_1$ does not affect the final result because the wave functions Ψ, Ψ' are determined up to a sign. Direct calculation of the Hamilton operator in the FW representation also leads to Eq. (17) in accordance with formulae (2),(3),(14),(18).

Another class of Hamiltonians permitting exact FW transformation has been investigated in [26].

V. EXACT TRANSFORMATION FOR PARTICLES IN ELECTROWEAK FIELDS

Let us consider the interaction of a relativistic spin-1/2 particle, possessing an anomalous magnetic moment (AMM), with stationary electromagnetic and electroweak fields. The Hamiltonian of the electromagnetic interaction is defined by the Dirac-Pauli equation [30]:

$$\mathcal{H}_{DP} = \boldsymbol{\alpha}\cdot\boldsymbol{\pi} + \beta m + e\Phi + \mu'(-\boldsymbol{\Pi}\cdot\boldsymbol{H} + i\boldsymbol{\gamma}\cdot\boldsymbol{E}), \qquad \boldsymbol{\pi} = \boldsymbol{p} - e\boldsymbol{A}, \tag{19}$$

where μ' is AMM, $\Phi, \boldsymbol{A}$ and $\boldsymbol{E}, \boldsymbol{H}$ are the potentials and the strengths of an electromagnetic field. This equation is derived in the one-particle approximation and is useful when an electromagnetic field is not extremely strong (see [11]).

The weak interaction Hamiltonian should be added to the Hamiltonian of Eq. (19). The weak interaction does not conserve the spatial parity. For the interaction transferred by neutral currents, the standard model gives the following expression for the parity-nonconserving

weak interaction Hamiltonian in the approximation of a small transferred momentum [31]:

$$\mathcal{H}_{PNC}=-\frac{G}{\sqrt{2}}\left(C_1\gamma^5+C_2\boldsymbol{\alpha}\cdot\boldsymbol{\sigma}'\right)n(\boldsymbol{r}),\quad \gamma^5=\left(\begin{array}{cc}0&-1\\-1&0\end{array}\right),\tag{20}$$

where G is the Fermi constant, $\boldsymbol{\sigma}'$ is the Pauli matrix for matter particles, and $n(\boldsymbol{r})$ is the density of matter particles. For the interactions with nuclei, $n(\boldsymbol{r})$ characterizes the density of nucleons of a certain kind, and $\boldsymbol{\sigma}'$ should be replaced by the nucleus spin. Formulae (19),(20) do not change if the external fields are nonstationary. The matter particles are considered to be at rest.

The coefficients C_1, C_2 are different for different pairs of interacting particles. The Hamiltonians corresponding to the interactions with different matter particles should be summed-up. The signs in formula (20) depend on the definition of the coefficients C_1, C_2 and matrix γ^5. The total Hamiltonian equals

$$\mathcal{H}=\mathcal{H}_{DP}+\mathcal{H}_{PNC}.\tag{21}$$

In this case, in formulae (3),(16)–(18) we have

$$\begin{array}{c}\mathcal{E}=e\Phi-\mu'\boldsymbol{\Pi}\cdot\boldsymbol{H},\qquad \mathcal{O}=\boldsymbol{\alpha}\cdot\boldsymbol{\pi}+i\mu'\boldsymbol{\gamma}\cdot\boldsymbol{E}-\frac{G}{\sqrt{2}}\left(C_1\gamma^5+C_2\boldsymbol{\alpha}\cdot\boldsymbol{\sigma}'\right)n(\boldsymbol{r})\\ =\beta\left[\boldsymbol{\gamma}\cdot\boldsymbol{\pi}+i\mu'\boldsymbol{\alpha}\cdot\boldsymbol{E}-\frac{G}{\sqrt{2}}\left(C_1\beta\gamma^5+C_2\boldsymbol{\gamma}\cdot\boldsymbol{\sigma}'\right)n(\boldsymbol{r})\right].\end{array}\tag{22}$$

Let us consider some particular cases where Hamiltonian (21) satisfies condition (14). For these cases the FW transformation is exact. The general case will be analyzed in the next section.

The exact Hamiltonian in the FW representation is given by Eq. (17), where $\mathcal{E}$ is defined by Eq. (22), and

$$\begin{array}{c}\epsilon=\left\{m^2-\left[\boldsymbol{\gamma}\cdot\boldsymbol{\pi}+i\mu'\boldsymbol{\alpha}\cdot\boldsymbol{E}-\frac{G}{\sqrt{2}}\left(C_1\beta\gamma^5+C_2\boldsymbol{\gamma}\cdot\boldsymbol{\sigma}'\right)n(\boldsymbol{r})\right]\right\}^{1/2}\\ =\left\{m^2+\boldsymbol{\pi}^2+\beta\mu'\left(\boldsymbol{\Sigma}\cdot[\boldsymbol{\pi}\times\boldsymbol{E}]-\boldsymbol{\Sigma}\cdot[\boldsymbol{E}\times\boldsymbol{\pi}]-\nabla\cdot\boldsymbol{E}\right)+\mu'^2\boldsymbol{E}^2-e\boldsymbol{\Sigma}\cdot\boldsymbol{H}\right.\\ +\frac{G}{\sqrt{2}}\Bigl(C_1\{\boldsymbol{\Sigma}\cdot\boldsymbol{\pi},n(\boldsymbol{r})\}_+-C_2\{\boldsymbol{\sigma}'\cdot\boldsymbol{\pi},n(\boldsymbol{r})\}_++C_2[\boldsymbol{\Sigma}\times\boldsymbol{\sigma}']\cdot\nabla n(\boldsymbol{r})\\ \left.-2\beta\mu'C_2[\boldsymbol{\Sigma}\times\boldsymbol{\sigma}']\cdot\boldsymbol{E}n(\boldsymbol{r})\Bigr)+\frac{G^2}{2}n^2(\boldsymbol{r})\left[C_1^2+3C_2^2-2C_2(C_1+C_2)\boldsymbol{\Sigma}\cdot\boldsymbol{\sigma}'\right]\right\}^{1/2}.\end{array}\tag{23}$$

Hence, the operator $U^{\pm}$ is expressed by formulae (18),(22),(23). Although Eq. (23) is formally exact, the small terms proportional to C_1^2, C_1C_2, C_2^2 are wittingly negligible in the approximation of a small transferred momentum.

Formulae (17),(22),(23) describe the exact Hamilton operator in the FW representation in the following particular cases:

a) in the presence of only weak interaction ($\Phi=0, \boldsymbol{A}=0, \boldsymbol{E}=0, \boldsymbol{H}=0$);

b) for Dirac particles ($\mu'=0$) in magnetic and weak fields ($\Phi=0, \boldsymbol{E}=0$);

c) for uncharged particles with AMM in electric and weak fields ($e=0, \boldsymbol{A}=0, \boldsymbol{H}=0$);

d) for particles with AMM moving in the plane orthogonal to a static uniform magnetic field ($\Phi=0, \boldsymbol{E}=0, P_z=0, C_1=C_2=0$);

e) for uncharged particles with AMM moving in the plane orthogonal to a static uniform magnetic field. A static electric field (possibly nonuniform) is also orthogonal to the magnetic field ($e=0, \boldsymbol{E}\perp\boldsymbol{H}, P_z=0, C_1=C_2=0$).

In two cases (d) and e)), $\boldsymbol{H}=H\boldsymbol{e}_z$, and in the case e), the electric field strength does not depend on z. Otherwise, rot$\boldsymbol{E}\neq 0$ and the magnetic field is not constant ($\partial\boldsymbol{H}/\partial t\neq 0$). Therefore, in these cases the operator $p_z=-i(\partial/\partial z)$ commutes with the Hamilton operator and has eigenvalues $P_z=$const. Consequently, the consideration of the particular case $P_z=0$ is quite reasonable. All these cases satisfy condition (14).

Formulae (17),(22),(23) agree with all exact expressions of the operator $\mathcal{H}'$ derived for uncharged particles with AMM in an electrostatic field, Dirac particles in a static magnetic field, and particles with AMM moving in the plane orthogonal to a static uniform magnetic field in [5, 7, 8]. The weak interaction is not considered in these works.

VI. GENERAL CASE

In the general case, relativistic particles interact with external fields. We suggest to perform the FW transformation in two stages. First, a transformation similar to the FW transformation for free particles, is performed for particles in external fields. Second, a transformation similar to the FW transformation for nonrelativistic particles is carried out.

We assume that the external fields are not extremely strong and the transformed Hamiltonian can be expressed as a power series in the field potentials and their derivatives. The external fields can be nonstationary.

In the general case, formula (17) is not exact because $\mathcal{E}$ depends on the coordinates and contains Dirac matrices. We should calculate the commutator of the operators U and $\mathcal{E}-i(\partial/\partial t)$:

$$U\left(\mathcal{E}-i\frac{\partial}{\partial t}\right)U^{-1}=\mathcal{E}-i\frac{\partial}{\partial t}+\left[U,\mathcal{E}-i\frac{\partial}{\partial t}\right]U^{-1}.$$

In this case, it is necessary to compute some commutators containing inverse operators and square roots of the operators. These commutators can be calculated using the following exact formulae which are valid for arbitrary operators A and B [32]:

$$[A^{-1},B]=A^{-1}[B,A]A^{-1}, \tag{24}$$

$$[A,B]=\frac{1}{4}\{A^{-1},[A^2,B]\}_+-\frac{1}{4}\left[[A,[A,B]],A^{-1}\right], \tag{25}$$

where $A^{-1}\equiv 1/A$ and $\{\dots,\dots\}_+$ stands for the anticommutator. If A is the square root of the operators and the commutator of the operators is small compared to their product, i.e.,

$$|[A,B]|\ll|AB|,$$

formulae (24),(25) allow us to obtain the quantity $[A,B]$ with any accuracy by the method of successive approximations (see [32]). As a rule, this condition is satisfied since it is equivalent to the inequality

$$\frac{\hbar c}{E}\ll l_c, \tag{26}$$

where$E$ is the total energy including the rest energy and l_c is the characteristic size of the nonuniformity region of the external field. For the nonrelativistic particle, the quantity $\hbar c/E$ is equal to the Compton wavelength.

First, it is necessary to perform a unitary transformation with operator (18). After this operation, the Hamiltonian $\mathcal{H}'$ still contains odd terms proportional to the derivatives of the potentials. Let us write the operator $\mathcal{H}'$ as

$$\mathcal{H}'=\beta\epsilon+\mathcal{E}'+\mathcal{O}',\quad \beta\mathcal{E}'=\mathcal{E}'\beta,\quad \beta\mathcal{O}'=-\mathcal{O}'\beta, \tag{27}$$

where

$$\begin{gathered}\epsilon=\sqrt{m^2+\mathcal{O}^2},\\ \mathcal{E}'=i\frac{\partial}{\partial t}+\frac{\epsilon+m}{\sqrt{2\epsilon(\epsilon+m)}}\left(\mathcal{E}-i\frac{\partial}{\partial t}\right)\frac{\epsilon+m}{\sqrt{2\epsilon(\epsilon+m)}}-\frac{\beta\mathcal{O}}{\sqrt{2\epsilon(\epsilon+m)}}\left(\mathcal{E}-i\frac{\partial}{\partial t}\right)\frac{\beta\mathcal{O}}{\sqrt{2\epsilon(\epsilon+m)}},\\ \mathcal{O}'=\frac{\beta\mathcal{O}}{\sqrt{2\epsilon(\epsilon+m)}}\left(\mathcal{E}-i\frac{\partial}{\partial t}\right)\frac{\epsilon+m}{\sqrt{2\epsilon(\epsilon+m)}}-\frac{\epsilon+m}{\sqrt{2\epsilon(\epsilon+m)}}\left(\mathcal{E}-i\frac{\partial}{\partial t}\right)\frac{\beta\mathcal{O}}{\sqrt{2\epsilon(\epsilon+m)}}.\end{gathered} \tag{28}$$

Since

$$ABA = \frac{1}{2}\left(\{A^2, B\}_+ - [A, [A, B]]\right),$$

relation (28) for the operator $\mathcal{E}'$ takes the form

$$\begin{aligned}\mathcal{E}' = \mathcal{E} - \frac{1}{4}\left[\frac{\epsilon+m}{\sqrt{2\epsilon(\epsilon+m)}}, \left[\frac{\epsilon+m}{\sqrt{2\epsilon(\epsilon+m)}}, \left(\mathcal{E} - i\frac{\partial}{\partial t}\right)\right]\right] \\ +\frac{1}{4}\left[\frac{\beta\mathcal{O}}{\sqrt{2\epsilon(\epsilon+m)}}, \left[\frac{\beta\mathcal{O}}{\sqrt{2\epsilon(\epsilon+m)}}, \left(\mathcal{E} - i\frac{\partial}{\partial t}\right)\right]\right].\end{aligned} \tag{29}$$

The odd terms are small compared to both ϵ and the initial Hamiltonian $\mathcal{H}$. This circumstance allows us to apply the usual scheme of the nonrelativistic FW transformation [1, 20].

Second, the transformation should be performed with the following operator:

$$U' = \exp(iS'), \quad S' = -\frac{i}{4}\beta\left\{\mathcal{O}', \frac{1}{\epsilon}\right\}_+ = -\frac{i}{4}\left[\frac{\beta}{\epsilon}, \mathcal{O}'\right]. \tag{30}$$

The further calculations are similar to those given in [20]. The particle mass should be replaced by the operator ϵ noncommuting with the operators $\mathcal{E}', \mathcal{O}'$. If only major corrections are taken into account, then the transformed Hamiltonian equals

$$\mathcal{H}'' = \beta\epsilon + \mathcal{E}' + \frac{1}{4}\beta\left\{\mathcal{O}'^2, \frac{1}{\epsilon}\right\}_+. \tag{31}$$

This is the Hamiltonian in the FW representation.

To obtain the desired accuracy, the calculation procedure with the transformation operator (30) (S' is replaced by S'', S''' *etc.*) should be repeated multiply.

Let us calculate the Hamiltonian in the FW representation for the relativistic particle with AMM interacting with a nonstationary electroweak field. The Hamiltonian in the Dirac representation is defined by formulae (19)–(21). The transformed Hamiltonian is defined by Eq. (30), where the operator $\mathcal{O}'$ contains the field strengths and does not contain the field potentials. Let us deduce the Hamiltonian to within first-order terms in the field strengths and their first derivatives and second-order terms in the field potentials. The terms of the second order and higher in the field strengths and their derivatives and the first-order terms containing derivatives of the second order and higher of the field strengths will be omitted.

Since we neglect the second-order quantities in $\mathcal{O}'$, the operator $\mathcal{O}'$ does not make any contribution to the Hamiltonian $\mathcal{H}''$ at the second stage of transformation defined by formula (31). As a result, we obtain the following equation for the Hamiltonian in the FW

representation:

$$
\begin{aligned}
\mathcal{H}'' &= \beta\epsilon + \mathcal{E}', \\
\mathcal{E}' &= e\Phi + \frac{e}{8}\left\{\frac{1}{\epsilon(\epsilon+m)}, (\boldsymbol{\Sigma}\cdot[\boldsymbol{\pi}\times\boldsymbol{E}] - \boldsymbol{\Sigma}\cdot[\boldsymbol{E}\times\boldsymbol{\pi}] - \nabla\cdot\boldsymbol{E})\right\}_+ \\
&+ \frac{e}{32}\left\{\frac{2\epsilon^2+2\epsilon m+m^2}{\epsilon^4(\epsilon+m)^2}, \boldsymbol{\pi}\cdot\nabla(\boldsymbol{\pi}\cdot\boldsymbol{E}+\boldsymbol{E}\cdot\boldsymbol{\pi})\right\}_+ - \mu'\boldsymbol{\Pi}\cdot\boldsymbol{H} \\
&+ \beta\frac{\mu'}{4}\left\{\frac{1}{\epsilon(\epsilon+m)}, \Big[(\boldsymbol{H}\cdot\boldsymbol{\pi})(\boldsymbol{\Sigma}\cdot\boldsymbol{\pi}) + (\boldsymbol{\Sigma}\cdot\boldsymbol{\pi})(\boldsymbol{\pi}\cdot\boldsymbol{H}) + 2\pi(\boldsymbol{\pi}\cdot\boldsymbol{j}+\boldsymbol{j}\cdot\boldsymbol{\pi})\Big]\right\}_+,
\end{aligned}
\tag{32}
$$

where $\boldsymbol{j} = \nabla\times\boldsymbol{H} - \dfrac{1}{4\pi}\dfrac{\partial\boldsymbol{E}}{\partial t}$ is the external current density, and ϵ is determined by formulae (23),(27). It is important that the operators $\epsilon, \mathcal{E}'$ are found at the first stage, i.e., at the transformation with operator (18).

In the weak field approximation,

$$
\begin{aligned}
\epsilon = \epsilon' &+ \beta\frac{\mu'}{4}\left\{\frac{1}{\epsilon'}, \Big(\boldsymbol{\Sigma}\cdot[\boldsymbol{\pi}\times\boldsymbol{E}] - \boldsymbol{\Sigma}\cdot[\boldsymbol{E}\times\boldsymbol{\pi}] - \nabla\cdot\boldsymbol{E}\Big)\right\}_+ \\
&- \frac{e}{4}\left\{\frac{1}{\epsilon'}, \boldsymbol{\Sigma}\cdot\boldsymbol{H}\right\}_+ + \frac{G}{4\sqrt{2}}\left\{\frac{1}{\epsilon'}, W\right\}_+
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{H}'' = \beta\epsilon' + e\Phi &+ \frac{1}{4}\left\{\left(\frac{\mu_0 m}{\epsilon'+m}+\mu'\right)\frac{1}{\epsilon'}, \Big(\boldsymbol{\Sigma}\cdot[\boldsymbol{\pi}\times\boldsymbol{E}] - \boldsymbol{\Sigma}\cdot[\boldsymbol{E}\times\boldsymbol{\pi}] - \nabla\cdot\boldsymbol{E}\Big)\right\}_+ \\
&+ \frac{\mu_0 m}{16}\left\{\frac{2\epsilon'^2+2\epsilon' m+m^2}{\epsilon'^4(\epsilon'+m)^2}, \boldsymbol{\pi}\cdot\nabla(\boldsymbol{\pi}\cdot\boldsymbol{E}+\boldsymbol{E}\cdot\boldsymbol{\pi})\right\}_+ - \frac{1}{2}\left\{\left(\frac{\mu_0 m}{\epsilon'}+\mu'\right), \boldsymbol{\Pi}\cdot\boldsymbol{H}\right\}_+ \\
+\frac{\mu'}{4}\left\{\frac{1}{\epsilon'(\epsilon'+m)}, \Big[(\boldsymbol{H}\cdot\boldsymbol{\pi})(\boldsymbol{\Pi}\cdot\boldsymbol{\pi}) + (\boldsymbol{\Pi}\cdot\boldsymbol{\pi})(\boldsymbol{\pi}\cdot\boldsymbol{H}) + 2\pi(\boldsymbol{\pi}\cdot\boldsymbol{j}+\boldsymbol{j}\cdot\boldsymbol{\pi})\Big]\right\}_+ &+ \frac{G}{4\sqrt{2}}\left\{\frac{1}{\epsilon'}, W\right\}_+,
\end{aligned}
\tag{33}
$$

where

$$
\epsilon' = \sqrt{m^2+\boldsymbol{\pi}^2}, \quad W = C_1\{\boldsymbol{\Sigma}\cdot\boldsymbol{\pi}, n(\boldsymbol{r})\}_+ - C_2\{\boldsymbol{\sigma}'\cdot\boldsymbol{\pi}, n(\boldsymbol{r})\}_+ + C_2[\boldsymbol{\Sigma}\times\boldsymbol{\sigma}']\cdot\nabla n(\boldsymbol{r}), \tag{34}
$$

and $\mu_0 = e/(2m)$ is the Dirac magnetic moment.

Unlike works [9, 10, 11], formula (33) includes, as particular cases, the exact expressions for the Hamiltonian in the FW representation obtained in [5, 7, 8]. Formulae (32)–(34) also agree with the results obtained in [9, 10, 11, 33, 34]. Detailed analysis shows that the method of FW transformation used in [11] does not allow one to take into consideration the terms proportional to the double commutators of ϵ with $e\Phi$ and $\mu'\boldsymbol{\Pi}\cdot\boldsymbol{H}$ $\big(e[\epsilon,[\epsilon,\Phi]]$ and $\mu'[\epsilon,[\epsilon,\boldsymbol{\Pi}\cdot\boldsymbol{H}]]\big)$. The terms proportional to $\boldsymbol{E}$ and $\boldsymbol{H}$ in the Hamiltonian obtained in [11] coincide with those in [9, 10] and Eq. (33). However, only the Dirac particles were considered

in [9], the derivatives of the field strengths were neglected in [10], and the nonrelativistic Hamiltonian with relativistic corrections was found in [33].

VII. PARTICLE AND SPIN MOTION EQUATIONS

Thus, the FW representation is very convenient for describing the particle and spin motion owing to the simple forms of operators. In order to derive corresponding quantum equations, it is necessary to compute the commutators of the Hamiltonian with the same operators as in the nonrelativistic theory. The kinetic momentum operator of particles in an electromagnetic field equals $\boldsymbol{\pi}=\boldsymbol{p}-e\boldsymbol{A}$ [35]. The equation of particle motion in the electromagnetic field is defined in terms of the commutator of the Hamiltonian with this operator:

$$\frac{d\boldsymbol{\pi}}{dt}=i[\mathcal{H}'',\boldsymbol{\pi}]-e\frac{\partial \boldsymbol{A}}{dt}.$$

To determine the quantum equation of particle motion, we take into account Eq. (33) for the Hamiltonian

$$\begin{aligned}\frac{d\boldsymbol{\pi}}{dt}&=e\boldsymbol{E}+\beta\frac{e}{4}\left\{\frac{1}{\epsilon'},\Bigl([\boldsymbol{\pi}\times\boldsymbol{H}]-[\boldsymbol{H}\times\boldsymbol{\pi}]\Bigr)\right\}_+\\&+\frac14\left\{\left(\frac{\mu_0 m}{\epsilon'+m}+\mu'\right)\frac{1}{\epsilon'},\Bigl[\nabla(\boldsymbol{\Sigma}\cdot[\boldsymbol{E}\times\boldsymbol{\pi}])-\nabla(\boldsymbol{\Sigma}\cdot[\boldsymbol{\pi}\times\boldsymbol{E}])+\Delta\boldsymbol{E}\Bigr]\right\}_+\\&-\frac{\mu_0 m}{16}\left\{\frac{2\epsilon'^2+2\epsilon' m+m^2}{\epsilon'^4(\epsilon'+m)^2},(\boldsymbol{\pi}\cdot\nabla)\nabla(\boldsymbol{\pi}\cdot\boldsymbol{E}+\boldsymbol{E}\cdot\boldsymbol{\pi})\right\}_+ +\frac12\left\{\left(\frac{\mu_0 m}{\epsilon'}+\mu'\right),\nabla(\boldsymbol{\Pi}\cdot\boldsymbol{H})\right\}_+\\&-\frac{\mu'}{4}\left\{\frac{1}{\epsilon'(\epsilon'+m)},\Bigl[(\boldsymbol{\Pi}\cdot\boldsymbol{\pi})\nabla(\boldsymbol{\pi}\cdot\boldsymbol{H})+\Bigl(\nabla(\boldsymbol{H}\cdot\boldsymbol{\pi})\Bigr)(\boldsymbol{\Pi}\cdot\boldsymbol{\pi})+2\pi\nabla(\boldsymbol{\pi}\cdot\boldsymbol{j}+\boldsymbol{j}\cdot\boldsymbol{\pi})\Bigr]\right\}_+.\end{aligned}\tag{35}$$

The equation of spin motion is defined by the formula

$$\frac{d\boldsymbol{\Pi}}{dt}=i[\mathcal{H}'',\boldsymbol{\Pi}].$$

For particles in a nonstationary electroweak field, it takes the form

$$\begin{aligned}\frac{d\boldsymbol{\Pi}}{dt}=&\left\{\left(\frac{\mu_0 m}{\epsilon'+m}+\mu'\right)\frac{1}{\epsilon'},[\boldsymbol{\Pi}\times[\boldsymbol{E}\times\boldsymbol{\pi}]]\right\}_+ +\left\{\left(\frac{\mu_0 m}{\epsilon'}+\mu'\right),[\boldsymbol{\Sigma}\times\boldsymbol{H}]\right\}_+\\&-\frac{\mu'}{2}\left\{\frac{1}{\epsilon'(\epsilon'+m)},\Bigl([\boldsymbol{\Sigma}\times\boldsymbol{\pi}](\boldsymbol{\pi}\cdot\boldsymbol{H})+(\boldsymbol{H}\cdot\boldsymbol{\pi})[\boldsymbol{\Sigma}\times\boldsymbol{\pi}]\Bigr)\right\}_+\\&-\frac{G}{2\sqrt2}\left\{\frac{1}{\epsilon'},\Bigl(C_1\left\{[\boldsymbol{\Sigma}\times\boldsymbol{\pi}],n(\boldsymbol{r})\right\}_+ +C_2\left[\boldsymbol{\Sigma}\times[\boldsymbol{\sigma'}\times\nabla n(\boldsymbol{r})]\right]\Bigr)\right\}_+.\end{aligned}\tag{36}$$

The corresponding equation for stationary electroweak fields was derived in [34].

The transition to the semiclassical description is also simple. For free particles, the lower spinor is equal to zero in the FW representation. For particles in external fields, the maximum ratio of the lower and upper spinors is of the first order of W_{int}/E, where W_{int} is the energy of the particle interaction with external fields. Thus, we obtain $(\chi^\dagger\chi)/(\phi^\dagger\phi) \sim (W_{int}/E)^2$. Therefore, the contribution of the lower spinor is negligible and the transition to the semiclassical equations is performed by averaging the operators in the equations for the upper spinor. It is usually possible to neglect the commutators between the coordinate and kinetic momentum operators and between different components of the kinetic momentum operator (see [36]). As a result, the operators $\boldsymbol{\sigma}, \boldsymbol{\sigma'}$ and $\boldsymbol{\pi}$ should be substituted by the corresponding classical quantities: the average spin, $\boldsymbol{\xi}$ ($\boldsymbol{\xi'}$ for matter particles), and the kinetic momentum. For the latter quantity we retain the designation $\boldsymbol{\pi}$. The semiclassical equations of particle and spin motion are:

$$
\begin{gathered}
\frac{d\boldsymbol{\pi}}{dt} = e\boldsymbol{E} + \frac{e}{\epsilon'}[\boldsymbol{\pi}\times\boldsymbol{H}] - \frac{1}{2}\left(\frac{\mu_0 m}{\epsilon'+m} + \mu'\right)\frac{1}{\epsilon'}\left[2\nabla(\boldsymbol{\xi}\cdot[\boldsymbol{\pi}\times\boldsymbol{E}]) - \Delta\boldsymbol{E}\right] \\
-\frac{\mu_0 m}{4}\cdot\frac{2\epsilon'^2 + 2\epsilon' m + m^2}{\epsilon'^4(\epsilon'+m)^2}(\boldsymbol{\pi}\cdot\nabla)\nabla(\boldsymbol{\pi}\cdot\boldsymbol{E}) + \left(\frac{\mu_0 m}{\epsilon'} + \mu'\right)\nabla(\boldsymbol{\xi}\cdot\boldsymbol{H}) \\
-\frac{\mu'}{\epsilon'(\epsilon'+m)}\left[(\boldsymbol{\xi}\cdot\boldsymbol{\pi})\nabla(\boldsymbol{H}\cdot\boldsymbol{\pi}) + 2\pi\nabla(\boldsymbol{j}\cdot\boldsymbol{\pi})\right],
\end{gathered} \tag{37}
$$

$$
\begin{gathered}
\frac{d\boldsymbol{\xi}}{dt} = 2\left(\frac{\mu_0 m}{\epsilon'+m} + \mu'\right)\frac{1}{\epsilon'}\left[\boldsymbol{\xi}\times[\boldsymbol{E}\times\boldsymbol{\pi}]\right] + 2\left(\frac{\mu_0 m}{\epsilon'} + \mu'\right)[\boldsymbol{\xi}\times\boldsymbol{H}] \\
-\frac{2\mu'}{\epsilon'(\epsilon'+m)}(\boldsymbol{H}\cdot\boldsymbol{\pi})[\boldsymbol{\xi}\times\boldsymbol{\pi}] - \frac{G}{\sqrt{2}\epsilon'}\left(2C_1[\boldsymbol{\xi}\times\boldsymbol{\pi}]n(\boldsymbol{r}) + C_2\left[\boldsymbol{\xi}\times[\boldsymbol{\xi'}\times\nabla n(\boldsymbol{r})]\right]\right).
\end{gathered} \tag{38}
$$

Equation (37) shows that the particle motion depends on the spin orientation. The corresponding term determines the Stern-Gerlach force.

It is not as convenient to use the Dirac representation to derive quantum equations of particle and spin motion in a similar manner. In this case, it is necessary to extract the polarization operator, $\boldsymbol{O}$, from the obtained equations. This problem is rather difficult because the operator $\boldsymbol{O}$ in the Dirac representation is defined by the cumbersome expression given in [17].

For particles in external fields, the FW transformation also changes the form of the kinetic momentum operator. In particular, the equation of spin motion in the Dirac representation

depends on the operator $\boldsymbol{\pi}_D = U^{-1}\boldsymbol{\pi}U$ just as the corresponding equation in the FW representation depends on the operator $\boldsymbol{\pi}$. However, these two equations differ in their functional dependence on $\boldsymbol{\pi}$. The use of the FW representation protects from both an error in derived equations of particle and spin motion and an incorrect interpretation of these equations.

Another method of transition to the semiclassical description is based on the trajectory-coherent solution of the Dirac equation [37].

VIII. DISCUSSION

As mentioned above, the proposed method permits obtaining a transformed Hamiltonian to within first-order terms in the field parameters after the first transformation. In this case, all the other canonical methods need several transformations [1, 5, 6, 10, 12, 20, 21]. Therefore, the method described above can be successfully used even for solving nonrelativistic problems. For this purpose, the transformation operator, U, can be expanded in a series of $1/m$. Of course, such an expansion is helpful only in the case of transformation of the operator $\mathcal{E} - i(\partial/\partial t)$. The transformation of other operators leads to the appearance of the term $\beta\epsilon$ in Eq. (31).

Consider the classical example of the FW transformation for a nonrelativistic Dirac particle in an electromagnetic field. We calculate the Hamiltonian to within terms of orders of $(p/m)^4$ and p^2W/m^3, where W means $e\Phi,\ e\boldsymbol{A}$ (see [1, 20]). In this approximation, the first double commutator in Eq. (29) is negligible and the second one is equal to

$$\begin{gathered}\frac{1}{4}\left[\frac{\beta\mathcal{O}}{\sqrt{2\epsilon(\epsilon+m)}},\left[\frac{\beta\mathcal{O}}{\sqrt{2\epsilon(\epsilon+m)}},\left(\mathcal{E}-i\frac{\partial}{\partial t}\right)\right]\right]\\ =\frac{1}{8m^2}\left[\boldsymbol{\gamma}\cdot\boldsymbol{\pi},\left[\boldsymbol{\gamma}\cdot\boldsymbol{\pi},\left(\mathcal{E}-i\frac{\partial}{\partial t}\right)\right]\right]=\frac{e}{8m^2}\left[\boldsymbol{\Sigma}\cdot(\boldsymbol{\pi}\times\boldsymbol{E})-\boldsymbol{\Sigma}\cdot(\boldsymbol{E}\times\boldsymbol{\pi})-\nabla\cdot\boldsymbol{E}\right].\end{gathered} \tag{39}$$

As

$$\epsilon=\sqrt{m^2+(\boldsymbol{\alpha}\cdot\boldsymbol{\pi})^2}=\sqrt{m^2+\boldsymbol{\pi}^2-e\boldsymbol{\Sigma}\cdot\boldsymbol{H}}=m+\frac{\boldsymbol{\pi}^2}{2m}-\frac{\boldsymbol{\pi}^4}{8m^3}-\frac{e}{2m}\boldsymbol{\Sigma}\cdot\boldsymbol{H},$$

the transformed Hamiltonian is expressed by the well-known formula [1, 20]:

$$\begin{aligned}\mathcal{H}''=\beta\left(m+\frac{\boldsymbol{\pi}^2}{2m}-\frac{\boldsymbol{\pi}^4}{8m^3}\right)+e\Phi-\frac{e}{2m}\boldsymbol{\Pi}\cdot\boldsymbol{H}\\ +\frac{e}{8m^2}\left[\boldsymbol{\Sigma}\cdot(\boldsymbol{\pi}\times\boldsymbol{E})-\boldsymbol{\Sigma}\cdot(\boldsymbol{E}\times\boldsymbol{\pi})-\nabla\cdot\boldsymbol{E}\right].\end{aligned} \tag{40}$$

Thus, the proposed method permits obtaining this formula after the computation of only one double commutator. All the other canonical methods require cumbersome calculations [1, 5, 12, 20, 21]. For example, the classical method of Foldy and Wouthuysen require three successive transformations and a calculation of numerous commutators. The noncanonical methods (the Pauli's elimination method and others) [4, 11, 21, 22, 23, 24, 25] permit deriving Hamiltonian (40) in an easier way. Nevertheless, the proposed method is very simple even compared to them. Moreover, it gives an opportunity to find a transformed Hamiltonian with any accuracy even for relativistic particles in external fields.

IX. SUMMARY

In this work, a method of FW transformation for relativistic particles in external fields is proposed. This method is simple and reliable. It performs the exact FW transformation as in known particular cases [5, 7, 8], as in others. This property distinguishes the proposed method from the other methods developed for relativistic particles [9, 10, 11]. The method is based on the well-known elaborations [1, 20]. First, a transformation similar to the FW transformation for free particles is performed for particles in external fields. Second, a transformation similar to the FW transformation for nonrelativistic particles is carried out. In the general case, the FW transformation is approximate. As an example, the Hamilton operator in the FW representation for relativistic particles with AMM interacting with nonstationary electroweak fields is found to within second derivatives of potentials.

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